Sliding Mode Control for Integrator Systems

part 2: Second and High Order Sliding Mode Control

Contents

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1 Notations of sgn and sig

In homogeneous theory and high order sliding mode, the **signum** function ensures the definition on $\mathbb{R}$ and the function is **odd**. We always use following notations with scalar value x

$$\operatorname{sgn}(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \qquad \lfloor x \rceil^q = |x|^q \operatorname{sgn}(x). \tag{1}$$

For vector $\boldsymbol{x} = [x_1, x_2, ..., x_n]^T \in \mathbb{R}^n$, two notations are used in literatures. The first one is a element-wise notation as

$$\lfloor \boldsymbol{x} \rceil^q = [|x_1|^q \operatorname{sgn}(x_1), |x_2|^q \operatorname{sgn}(x_2), ..., |x_n|^q \operatorname{sgn}(x_n)]^T \tag{2}$$

Most results of $\mathbb{R}^1$ can be used in this definition. However, this notation needs every controller part knows the global coordinate. So a notation inspired by unit-vector is also very famous, which is

$$\operatorname{sgn}(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } \|x\| \neq 0 \\ 0 & \text{if } \|x\| = 0 \end{cases}, \qquad \lfloor \boldsymbol{x} \rceil^q = \|\boldsymbol{x}\|^q \operatorname{sgn}(\boldsymbol{x}) \tag{3}$$

This operation has following properties:

1. derivative: $\frac{d}{dx} \lfloor x \rceil^q = q|x|^{q-1}$
2. integral: $\frac{d}{dx} |x|^q = q \lfloor x \rceil^{q-1}$
3. When power $q = 0$, $\lfloor x \rceil^0 = \operatorname{sgn}(x)$

For the vector case, when we use the second definition with Euclidean norm, the second propery is not satisfied.

1.1 Derivative and Integral of signum

If we use the Euclidean norm and the second definition of the signum function, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}\|\boldsymbol{x}\| = \frac{\partial}{\partial x_i}\sqrt{\sum_{i=1}^{n} x_i^2} = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2\right)^{-\frac{1}{2}} \frac{\partial}{\partial x_i}\sum_{i=1}^{n} x_i^2 = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2\right)^{-\frac{1}{2}} 2x_i = \frac{x_i}{\|\boldsymbol{x}\|} \\
\frac{\partial}{\partial \boldsymbol{x}}\|\boldsymbol{x}\| = \left[\frac{\partial}{\partial x_1}\|\boldsymbol{x}\|, \frac{\partial}{\partial x_2}\|\boldsymbol{x}\|, ..., \frac{\partial}{\partial x_n}\|\boldsymbol{x}\|\right] = \frac{\boldsymbol{x}^T}{\|\boldsymbol{x}\|} = (\operatorname{sgn}(\boldsymbol{x}))^T \\
\frac{\partial}{\partial \boldsymbol{x}}\lfloor \boldsymbol{x} \rceil^0 = \frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x}\|\boldsymbol{x}\| = \left[\frac{\partial}{\partial x_1}\|\boldsymbol{x}\|, \frac{\partial}{\partial x_2}\|\boldsymbol{x}\|, ..., \frac{\partial}{\partial x_n}\|\boldsymbol{x}\|\right] = \frac{\boldsymbol{x}^T}{\|\boldsymbol{x}\|}
\end{aligned} \tag{4}$$

Generally, when use the Euclidean norm, We can calculate the derivatives as

$$\frac{\partial}{\partial \boldsymbol{x}}\|\boldsymbol{x}\|^q = q * \|\boldsymbol{x}\|^{q-1}\frac{\partial}{\partial \boldsymbol{x}}\|\boldsymbol{x}\| = q * \|\boldsymbol{x}\|^{q-1}(\operatorname{sgn}(\boldsymbol{x}))^T = q\left(\lfloor \boldsymbol{x} \rceil^{q-1}\right)^T \tag{5}$$

Let $\boldsymbol{x} = \boldsymbol{x}(t)$ and the derivative of signum is

$$\frac{d}{dt}\lfloor \boldsymbol{x} \rceil^q = \frac{d}{dt}\|\boldsymbol{x}\|^{q-1} \cdot \boldsymbol{x} = \left((q-1)\lfloor \boldsymbol{x} \rceil^{q-2}\right)^T \cdot \dot{\boldsymbol{x}} \cdot \boldsymbol{x} + \|\boldsymbol{x}\|^{q-1} \cdot \dot{\boldsymbol{x}} = (q-1)\|\boldsymbol{x}\|^{q-1}\left((\operatorname{sgn}\boldsymbol{x})^T \dot{\boldsymbol{x}}\right) \operatorname{sgn}(\boldsymbol{x}) + \|\boldsymbol{x}\|^{q-1} \cdot \dot{\boldsymbol{x}} \tag{6}$$

In the scalar case,

$$\frac{d}{dt}\lfloor x \rceil^q = (q-1)\lfloor x \rceil^{q-2} \cdot \dot{x} \cdot x + \|x\|^{q-1} \cdot \dot{x} = q\|x\|^{q-1} \cdot \dot{x} \tag{7}$$

2 Super Twisting Control

The STA (Super Twisting Algorithm) can be written as

$$\begin{aligned} \dot{x}_1 &= -k_1 \lfloor x_1 \rceil^{\frac{1}{2}} + x_2 + \varrho_1(x,t) \\ \dot{x}_2 &= -k_2 \ \mathrm{sgn}(x_1) + \varrho_2(x,t) \end{aligned} \tag{8}$$

where x_i are the scalar state variables, k_i are gains to be designed, and ϱ_i are the perturbation terms. **Under some conditions on k_i, the algorithm is robust against a bounded perturbation** $\varrho_1(x,t) = 0$, $|\varrho_2(x,t)| \leq L$. Since the righthand side of (8) is discontinuous, the solutions will be understood in the sense of **Filippov**.

Finite time convergence and robustness for the STA has been proved by

1. **geometrical methods** (LEVANT, 2007)
2. **Homogeneity properties** of the algorithm (LEVANT, 2005; ORLOV, 2004)

- Weak Lyapunov function $V_{w(x)} = k_2|x_1| + \frac{1}{2}x_2^2$ (ORLOV, 2004), $\sqrt{V_{w(x)}}$ (UTKIN et al., 2017)
- Strong Lyapunov function: V in (POLYAKOV et al., 2009),(MORENO et al., 2012),(SEEBER et al., 2017)

"Weak" measns $\dot{V}_{w(x)} = -k_1 k_2 |x_1|^{\frac{1}{2}}$ is only negative semidefinite. (Finite time) convergence can only be asserted by using a generaization of LaSalle's invariance principle for discontinuous systems (ORLOV, 2004), but it is not possible to provide robustness results, or to estimate the convergence time from it. (UTKIN GULDNER SHI, 2017) analyse the weak Lyapunov function $\sqrt{V_{w(x)}}$ and the finite time and robust convergence for the STA is proved.

2.1 Strick Lyapunov Functions for STA

POLYAKOV A, POZNYAK A, 2009. Reaching time estimation for “super-twisting” second order sliding mode controller via lyapunov function designing[J/OL]. IEEE Transactions on Automatic Control, 54(8): 1951-1955[2024-04-11]. http://ieeexplore.ieee.org/document/5173492/. DOI:10.1109/TAC.2009.2023781.

$$V = \begin{cases} \frac{k^2}{4}\left(\frac{y\ \mathrm{sgn}(x_1)}{\gamma} + k_0 e^{m(x_1,x_2)}\sqrt{s(x_1,x_2)}\right)^2 & x_1 x_2 \neq 0 \\ \frac{2k^2 x_2^2}{\alpha^2} & x_1 = 0 \\ \frac{|x_1|}{2} & x_2 = 0 \end{cases} \tag{9}$$

MORENO J A, OSORIO M, 2012. Strict lyapunov functions for the super-twisting algorithm[J/OL]. IEEE Transactions on Automatic Control, 57(4): 1035-1040[2024-01-26]. https://ieeexplore.ieee.org/document/6144710. DOI:10.1109/TAC.2012.2186179.

$$V(x) = \zeta^T P \zeta, \zeta = \begin{bmatrix} \lfloor x_1 \rceil^{\frac{1}{2}} \\ x_2 \end{bmatrix} \tag{10}$$

SEEBER R, HORN M, 2017. Stability proof for a well-established super-twisting parameter setting[J/OL]. Automatica, 84: 241-243[2024-01-25]. https://linkinghub.elsevier.com/retrieve/pii/S000510981730328X. DOI:10.1016/j.automatica.2017.07.002.

$$V(x) = \begin{cases} 2\sqrt{x_2^2 + 3\alpha^2 k_1^2 x_1} - x_2 & x \in M \\ 2\sqrt{x_2^2 + 3\alpha^2 k_1^2 x_1} + x_2 & -x \in M \\ 3|x_2| & \text{otherwise} \end{cases} \tag{11}$$

3 Homogeneity

Homogeneous control laws appear as solutions to many control problems such as a minimum time feedback control for the chain of integrators or the high-order sliding mode design. The homogeneity allows some time constraints in control systems to be fulfilled by means of a proper selection of the so-called homogeneity degree. Similar to the linear case, an asymptotic stability of a homogeneous system implies its robustness (input-to-state stability) with respect to a certain class of parametric uncertainties and exogenous perturbations.

Many lliteratures like (POLYAKOV, 2020; □□□, 2010; SHTESSEL et al., 2014) talks about homogeneity.

Corollary 6.1 The global uniform finite-time stability of homogeneous differential equations (Filippov inclusions) with negative homogeneity degree is **robust with respect to locally small homogeneous perturbations**.

3.1 Homogeneity of coordinate, function and vector field

Assign a weight(the **homogeneity degree**) of each coordinate $x_i \in \mathbb{R}, i = 1, ..., n$, where $m_i > 0$. We will write

$$\deg(x_i) = m_i. \tag{12}$$

The corresponding simple linear transformation

$$d_k : (x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_n} x_n) \tag{13}$$

is called **homogeneity dilation**, and $\kappa > 0$ is called its parameter.

A **function** $f : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of the degree (weight) $q \in \mathbb{R}$ with the above homogeneity dilation and written as $\deg(f) = q$, if for any $\kappa > 0$, the identity $f(d_k x) = \kappa^q f(x)$ holds.

A **vector field** $f : \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, ..., f_n)^T$, is called homogeneous of degree $q \in \mathbb{R}$ with the above dilation and written as $\deg(f) = q$, if all its componenets f_i are homogeneous and the identities

$$\deg\ f_i = \deg\ x_i + \deg\ f = \deg\ x_i + q, i = 1, 2, ..., n \tag{14}$$

Let A and B be two homogeneous functions of $x \in \mathbb{R}^n$ different from identical zero, and let λ be a real number; then

1. The sum of A and B is a homogeneous function only if $\deg\ A = \deg\ B$
2. $\forall \lambda \neq 0$, we have $\deg \lambda = 0$
3. $\deg\ AB = \deg\ A + \deg\ B$
4. $\deg\ \frac{A}{B} = \deg\ A - \deg\ B$
5. $\deg\ \lambda A = \deg\ A$
6. $\deg\ \frac{\partial}{\partial x_i} A = \deg\ A - \deg\ x_i$ if $\frac{\partial}{\partial x_i} A$ is not identical zero

To verify the last equality it can be seen that

$$\begin{aligned} \frac{\partial}{\partial \kappa_i^m x_i} A(d_k x) &= \kappa^{-m_i} \frac{\partial}{\partial x_i} \kappa^{\deg\ A} A(x) \\ &= \kappa^{\deg\ A - m_i} \frac{\partial}{\partial x_i} A(x) \end{aligned} \tag{15}$$

The last equality tells that for a systme $\dot{x} = f(x)$, we have $\dot{x}_i = f_{i(x)}$. Then,

$$\deg\ \dot{x}_i = \deg\ f_i = \deg\ x_i - \deg t \tag{16}$$

3.2 Homogeneity of differential equations and inclusions

Take the one dimension system $\dot{x} = f(x) = x^2$ into cosideration, let deg $x = 1$, then the homogeneity degree of **function** f is deg $f = 2$ and the homogeneity degree of **vector field** f is deg $f = \deg f_i - \deg x_i = 1$. *The ambiguity disappears if we speak aout the homogeneity of the differential equation* $\dot{x} = f(x)$.

We call differential equation $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ homogeneous with degree q, if the system is invariant with respect to the linear time-coorinate transformation

$$G_k : (t, \boldsymbol{x}) \mapsto (\kappa^{-1} t, d_k x), \kappa > 0. \tag{17}$$

Then we have

$$f(x) = \kappa^{-1} d_k^{-1} f(d_k x) \tag{18}$$

. The homogeneity degree of the system is $\deg t = \deg f_i - \deg x_i = \deg \boldsymbol{f}$.

$$\begin{gathered} f_{i(d_k x)} = \kappa^{\deg f_i} f_{i(x)} = \kappa^{\deg x_i + \deg f} f_{i(x)} \\ \boldsymbol{f}(d_k x) = d_k \kappa^{\deg f} \boldsymbol{f}(x) \end{gathered} \tag{19}$$

Note: The nonzero homogeneity degree q of a vector field can always be scaled to ± 1 by an appropriate proportional change to the weights of the coordinates and time.

Definition 6.6 A vector-set field $F(x) \subset \mathbb{R}^n, x \in \mathbb{R}^n$, and the differential inclusion

$$\dot{x} \in F(x) \tag{20}$$

are called homogeneous of the degree $q \in \mathbb{R}$ with the dialtion, which is writen as deg $F = q$, if the DI is invariant with respect to the time-coordinate transformation (17)

3.3 Convergence Rates of Homogeneous Algorithms

3.3.1 Finite-Time and Fixed-Time Stabilization

Consider simplest scalar first order system

$$\dot{x} = u \tag{21}$$

- The classical approach fives the standard *linear* proportional feedback

$$u_{\text{linear}(x)} = -x \tag{22}$$

which guarantees an asymptotic(exponential) convergence to the origin of any trajectory of the closed-loop system: $|x(t)| = e^{-t}|x_0|$.

- The **globally homogeneous** feedback is

$$u_{\text{FT}(x)} = -\sqrt{|x|}\operatorname{sgn}(x). \tag{23}$$

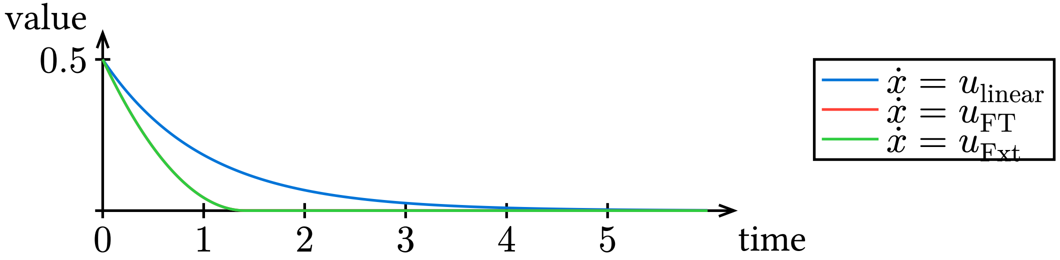
This algorithm stabilizes the system at the origin in a *finite time*:

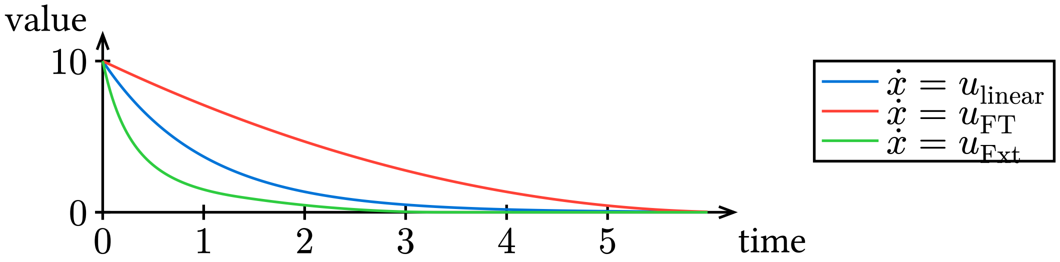
$$x(t) = 0, \text{for } t \geq T(x_0) = 2\sqrt{|x_0|} \tag{24}$$

- The *fixed-time* stabilizing controller can be selected **locally homogeneous** in the form:

$$u_{\text{FxT}(x)} = \begin{cases} -\lfloor x \rceil^{\frac{1}{2}} & |x| \leq 1 \\ -\lfloor x \rceil^{\frac{3}{2}} & |x| > 1 \end{cases}. \tag{25}$$

The system will be stabilized within 4 second, that is $x(t) = 0, \forall\ t \geq 4\ \forall x_0 \in \mathbb{R}$



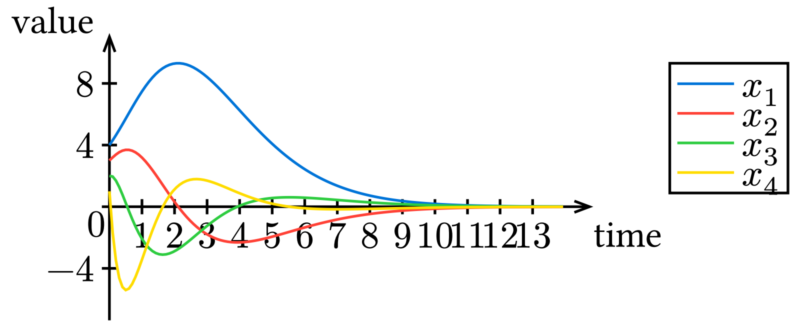
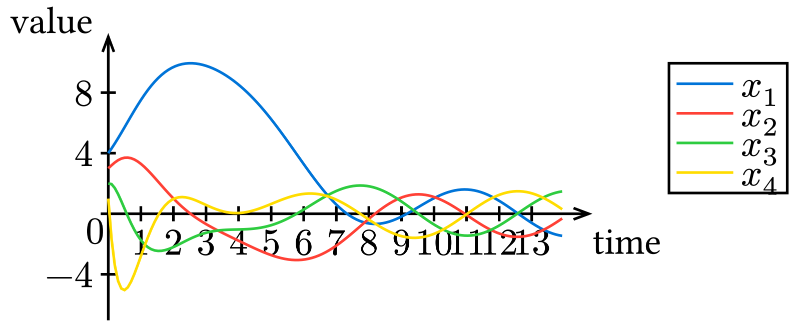
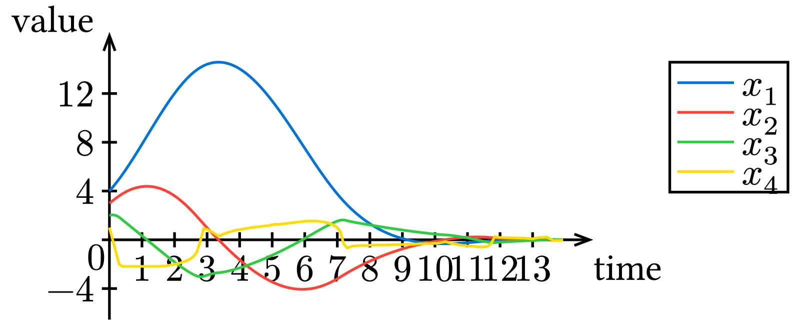
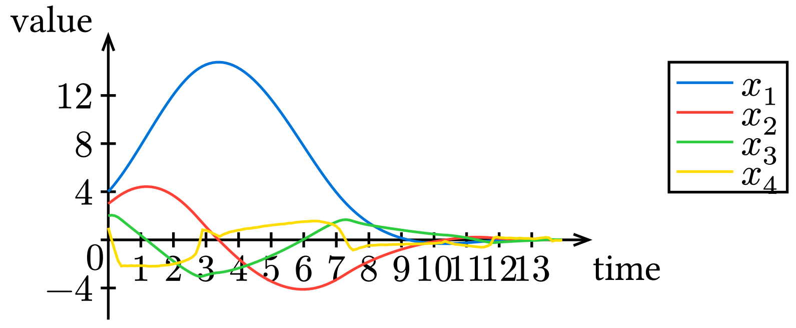


3.3.2 Robustness

3.3.3 Elimination of an Unbounded "Peaking" Effect

4 High-Order Sliding Mode Control for Integrator Systems

Tranditionally, a linear feedback can stabilize high-order system without robustness

	$x^{(4)} = u$	$x^{(4)} = u + 10 * \sin(t)$
linear feedback		
relay SMC feedback		

Consider the system

$$\sigma^{(r)} = u + \delta. \tag{26}$$

Nested Sliding Controllers are given by

$$\begin{aligned} u &= -\alpha \Psi_{r-1,r}\left(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}\right) \\ \Psi_{0,r} &= \operatorname{sign}(\sigma) \\ \Psi_{i,r} &= \operatorname{sign}\left(\sigma^{(i)} + \beta_i N_{i,r} \Psi_{i-1,r}\right) \\ N_{i,r} &= \left(|\sigma|^{\frac{1}{r}} + |\dot{\sigma}|^{\frac{q}{r-1}} + ... + |\sigma^{\frac{q}{r-i+1}}|\right)^{\frac{1}{q}} \end{aligned} \tag{27}$$

THANKS